TORSION UNITS FOR SOME UNTWISTED EXCEPTIONAL GROUPS OF LIE TYPE

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Abstract. In this paper, we investigate the Zassenhaus conjecture for exceptional groups of lie type $G_2(q)$ for $q = \{3, 4\}$. Consequently, we prove that the Prime graph question is true for these groups.

1. Introduction and main results

Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring of a finite group G. It is well known that

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G),$$

where $V(\mathbb{Z}G)$ is the group of units of augmentation one. G will represent a finite group throughout this article and torsion units will always represent torsion units in $V(\mathbb{Z}G) \setminus \{1\}$. The next conjecture is an important one in the theory of integral group rings is:

Conjecture 1. *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$ where $|u|$ and $|g|$ is the order of u and g respectively.*

Hans Zassenhaus formulated a stonger version of this onjecture in [37], which states that:

Conjecture 2. *A torsion unit in $V(\mathbb{Z}G)$ is said to be rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group algebra $\mathbb{Q}G$.*

This conjecture was confirmed for nilpotent groups in [32,36] and cyclic-by-abelian groups in [18]. The main investigative tool for simple groups G in relation to the Zassenhaus conjecture for $\mathbb{Z}G$ is the Luthar-Passi Method (which was introduced in [29]). It was confirmed true for all groups up to order 71, A_5, S_5, central extensions of S_5 and other simple finite groups in [4,5,25,29,30]. Partial results were given for A_6 in [33] and the remaining cases were dealt with in [23]. Alternating groups of higher order were also considered in [34,35]. It was also proved for $PSL(2,p)$ when $p = \{7, 11, 13\}$ in [24] and $PSL(2,p)$ where $p = \{8, 17\}$ in [21] and $PSL(2,p)$ when $p = \{19, 23\}$ in [2].

Let $t(H)$ be the torsion part of a group H of of finite exponent and $\#H$ be the set of primes dividing the order of elements from the set $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group H. The following, was composed as a problem in [31] (Problem 37):

Question 1. *(Prime Graph Question) If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

This question was upheld for Frobenius and Solvable groups in [27] and was also confirmed for some Sporadic Simple groups in [3,6–16]. The class of Chevalley groups $G_2(q)$ was first introduced by L. E. Dickson in [19] when q is odd and in [20] when q is even. Note that $G_2(q)$ has order $q^6(q^6-1)(q^2-1)$ and $G_2(q)$ is simple when $q > 2$. We use the Luthar-Passi Method to obtain our results. Our results are the following:

Theorem 1. *Let $G = G_2(3)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently, the following conditions hold:*

(*i*) *If $|u| \in \{2, 7, 13\}$, then u is rationally conjugate to some $g \in G$.*
(*ii*) *There are no elements of order 14, 21, 26, 39 or 91 in $V(\mathbb{Z}G)$.*

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Theorem 2. *Let $G = G_2(4)$ and u be a torsion unit of $V(\mathbb{Z}G)$. Subsequently, the following conditions hold:*

(i) *If $|u| \in \{7, 13\}$, then u is rationally conjugate to some $g \in G$.*

(ii) *There are no elements of order 14, 26, 35, 39, 65 or 91 in $V(\mathbb{Z}G)$.*

(iii) *If $|u| = 2$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{2a}, \nu_{2b}\}$ and*

$$(\nu_{2a}, \nu_{2b}) \in \{(4,-3), (3,-2), (2,-1), (1,0), (0,1), (-1,2), (-2,3), (-3,4)\}.$$

(iv) *If $|u| = 3$, then $\nu_{rx} = 0 \,\forall\, rx \notin \{\nu_{3a}, \nu_{3b}\}$ and*

$$(\nu_{3a}, \nu_{3b}) \in \{(1,0), (0,1), (-1,2)\}.$$

Consequently, we obtain the following result:

Corollary 1. *The Prime Graph question is true for the integral group ring of the groups $G_2(q)$ for $q \in \{3, 4\}$.*

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$. Then, the sum $\sum_{g \in X^G} a_g \in \mathbb{Z}$ which is the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to it's conjugacy classes X^G in G. Let $\nu_i = \varepsilon_{C_i}(u)$ be the i-th partial augmentation of u. It was proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central element by G. Higman and S. D. Berman [1]. Therefore $\nu_2 + \nu_3 + \cdots + \nu_l = 1$ where l denotes the number of non-central conjugacy classes of G.

Proposition 1. *([17]) Let u be a torsion unit of $V(\mathbb{Z}G)$. The order of u divides the exponent of G.*

The following Propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 2. *(Proposition 3.1 in [22]) Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation $\varepsilon_C(u) = 0$.*

Proposition 3. *(Proposition 2.2 in [24]) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

(i) *If u has order p^n, then $\varepsilon_x(u) = 0$ for every x of G whose p-part is of order strictly greater than p^n.*

(ii) *If x is an element of G whose p-part, for some prime, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Proposition 4. *([29]) Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ iff for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

For any character χ of G and any torsion unit u of $V(\mathbb{Z}G)$, clearly $\chi(u) = \sum_{i=2}^{l} \nu_i \chi(h_i)$ where h_i is a representative of a non-central conjugacy class C_i.

Proposition 5. *(Theorem 1 in [29] and [24]) Let p be equal to zero or a prime divisor of $|G|$. Suppose that u is an element of $V(\mathbb{Z}G)$ of order k. Let z be a primitive $k-th$ root of unity. Then for every integer l and any character χ of G, the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\}$$

is a non-negative integer.

We will use the notation $\mu_l(u, \chi, *)$ when $p = 0$. The LAGUNA package [28] for the GAP system [26] is a very useful tool when calculating $\mu_l(u, \chi, p)$.

2. Proof of Theorem 1

Let $G = G_2(4)$. Clearly $|G| = 4245696 = 2^6 \cdot 3^6 \cdot 7 \cdot 13$ and $exp(G) = 6552 = 2^3 \cdot 3^2 \cdot 7 \cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\begin{aligned}&\nu_{2a} + \nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{3d} + \nu_{3e} + \nu_{4a} + \nu_{4b} + \nu_{6a} + \nu_{6b} + \nu_{6c}\\ &\quad + \nu_{6d} + \nu_{7a} + \nu_{8a} + \nu_{8b} + \nu_{9a} + \nu_{9b} + \nu_{9c} + \nu_{12a} + \nu_{12b} + \nu_{13a} + \nu_{13b} = 1\end{aligned}$$

In order to prove that the Zassenhaus Conjecture holds, we need to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 18, 21, 24, 26, 39 and 91 (by Proposition 1). For the purpose of this paper and due to the complexity of certain orders, we shall consider elements of order 2, 7, 13, 14, 21, 26, 39 and 91. We shall now consider each case separately.

Case (i). Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. By Proposition 2, $\nu_{kx} = 0$ for all

$$kx \in \{3a, 3b, 3c, 3d, 3e, 4a, 4b, 6a, 6b, 6c, 6d, 7a, 8a, 8b, 9a, 9b, 9c, 12a, 12b, 13a, 13b\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (ii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 7$. By Proposition 2, $\nu_{kx} = 0$ for all

$$kx \in \{2a, 3a, 3b, 3c, 3d, 3e, 4a, 4b, 6a, 6b, 6c, 6d, 8a, 8b, 9a, 9b, 9c, 12a, 12b, 13a, 13b\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. Using Propositions 2 & 3,

$$\nu_{13a} + \nu_{13b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_1(u, \chi_{15}, *) = \tfrac{1}{13}(\gamma_1 + 448) \geq 0; \quad \mu_1(u, \chi_2, 3) = \tfrac{1}{13}(-\gamma_1 + 7) \geq 0;$$
$$\mu_2(u, \chi_2, 3) = \tfrac{1}{13}(\gamma_2 + 7) \geq 0$$

where $\gamma_1 = 7\nu_{13a} - 6\nu_{13b}$ and $\gamma_2 = 6\nu_{13a} - 7\nu_{13b}$. Clearly $\gamma_1 \in \{7 + 13k \mid -35 \leq k \leq 0\}$. It follows that the only possible integer solutions for (ν_{13a}, ν_{13b}) are $(0, 1)$ and $(1, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 14$. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{7a} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_7(u, \chi_2, *) = \tfrac{1}{14}(12\nu_{2a} + 16) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{14}(-12\nu_{2a} + 12) \geq 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{14}(-5\nu_{2a} + 96) \geq 0.$$

It follows that there are no possible integer solutions for (ν_{2a}, ν_{7a}).

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 21$. We shall first consider torsion units of $V(\mathbb{Z}G)$ of order 3. Using Propositions 2 & 3, $\nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{3d} + \nu_{3e} = 1$. Applying Proposition 5, we obtain the following system of inequalities:

$$\begin{aligned}
&\mu_0(u, \chi_2, *) = \tfrac{1}{3}(2\gamma_1 + 14) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{3}(-\gamma_1 + 14) \geq 0;\\
&\mu_1(u, \chi_3, *) = \tfrac{1}{3}(\gamma_2 + 64) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{3}(-2\gamma_2 + 64) \geq 0;\\
&\mu_1(u, \chi_5, *) = \tfrac{1}{3}(3\gamma_3 + 78) \geq 0; \quad \mu_0(u, \chi_5, *) = \tfrac{1}{3}(-6\gamma_3 + 78) \geq 0;\\
&\mu_1(u, \chi_7, *) = \tfrac{1}{3}(\gamma_4 + 91) \geq 0; \quad \mu_0(u, \chi_7, *) = \tfrac{1}{3}(-2\gamma_4 + 91) \geq 0;\\
&\mu_0(u, \chi_6, *) = \tfrac{1}{3}(\gamma_5 + 91) \geq 0; \quad \mu_0(u, \chi_8, *) = \tfrac{1}{3}(\gamma_6 + 91) \geq 0;\\
&\mu_1(u, \chi_8, *) = \tfrac{1}{3}(\gamma_7 + 91) \geq 0; \quad \mu_0(u, \chi_9, 7) = \tfrac{1}{3}(\gamma_8 + 103) \geq 0
\end{aligned}$$

where $\gamma_1 = 5\nu_{3a} + 5\nu_{3b} - 4\nu_{3c} + 2\nu_{3d} - \nu_{3e}$, $\gamma_2 = 8\nu_{3a} + 8\nu_{3b} - \nu_{3c} - 4\nu_{3d} + 2\nu_{3e}$, $\gamma_3 = \nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{3d} - 2\nu_{3e}$, $\gamma_4 = 8\nu_{3a} - 19\nu_{3b} - \nu_{3c} - 4\nu_{3d} + 2\nu_{3e}$, $\gamma_5 = 20\nu_{3a} + 20\nu_{3b} + 20\nu_{3c} + 2\nu_{3d} + 2\nu_{3e}$, $\gamma_6 = 38\nu_{3a} - 16\nu_{3b} + 2\nu_{3c} + 8\nu_{3d} - 4\nu_{3e}$, $\gamma_7 = -19\nu_{3a} + 8\nu_{3b} - \nu_{3c} - 4\nu_{3d} + 2\nu_{3e}$ and $\gamma_8 = 26\nu_{3a} + 26\nu_{3b} + 8\nu_{3c} + 2\nu_{3d} - 4\nu_{3e}$. Clearly $\gamma_1 \in \{2 + 3k \mid -3 \leq k \leq 4\}$, $\gamma_2 \in \{2 + 3k \mid -22 \leq k \leq 10\}$, $\gamma_3 \in \{k \mid -26 \leq k \leq 13\}$ and $\gamma_4 \in \{2 + 3k \mid -31 \leq k \leq 14\}$. It follows that there are 873 possible integer solutions for $(\nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{3d}, \nu_{3e})$. If we consider torsion units of order 21. There is only one partial augmentation for units of order 7 and 873 partial augmentations for units of order 3. Therefore we need to consider $873 \cdot 1 = 873$ cases for units of order 21. With the aid of LAGUNA ([28]) for the GAP system ([26]), we solved each case and it transpired that there are no solutions in each case.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{13a} + \nu_{13b} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(2a)$ and $\chi(u^2) = m_1\chi(13a)+m_1\chi(13b)$ where $(m_1, m_2) \in \{(1, 0), (0, 1)\}$. Applying Proposition 5, we obtain:

$$\mu_{13}(u, \chi_2, *) = \tfrac{1}{26}(12\gamma + 28) \geq 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{26}(-12\gamma + 24) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{26}(-\gamma + 15) \geq 0$$

where $\gamma = 2\nu_{2a} - \nu_{13a} - \nu_{13b}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{13a}, \nu_{13b})$.

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 39$. There are two partial augmentation for units of order 13 and 873 partial augmentations for units of order 3. Therefore we need to consider $873 \cdot 2 = 1746$ cases for units of order 21. With the aid of LAGUNA ([28]) for the GAP system ([26]), we solved case and it transpired that there are no solutions in each case.

Case $(viii)$. Let $u \in V(\mathbb{Z}G)$ where $|u| = 91$. Using Propositions 2 & 3,

$$\nu_{7a} + \nu_{13a} + \nu_{13b} = 1.$$

Consider the cases $\chi(u^{13}) = \chi(7a)$ and $\chi(u^7) = m_1\chi(13a)+m_1\chi(13b)$ where $(m_1, m_2) \in \{(1, 0), (0, 1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{91}(72\gamma + 26) \geq 0; \quad \mu_0(u, \chi_6, 3) = \tfrac{1}{91}(-216\gamma + 13) \geq 0$$

where $\gamma = \nu_{13a} + \nu_{13b}$. It follows that there are no possible integer solutions for $(\nu_{7a}, \nu_{13a}, \nu_{13b})$.

We shall now consider the prime graph of $G = G_2(4)$. G contains elements of order 6. Therefore $[2, 3]$ are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 14, 21, 26, 39 and 91 in $V(\mathbb{Z}G)$. This completes the proof.

3. Proof of Theorem 2

Let $G = G_2(4)$. Clearly $|G| = 251596800 = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $exp(G) = 10920 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13$. Initially, for any torsion unit of $V(\mathbb{Z}G)$ of order k:

$$\begin{aligned}&\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{3b} + \nu_{4a} + \nu_{4b} + \nu_{4c} + \nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d}\\ &\quad + \nu_{6a} + \nu_{6b} + \nu_{7a} + \nu_{8a} + \nu_{8b} + \nu_{10a} + \nu_{10b} + \nu_{10c} + \nu_{10d} + \nu_{12a}\\ &\quad\quad + \nu_{12b} + \nu_{12c} + \nu_{13a} + \nu_{13b} + \nu_{15a} + \nu_{15b} + \nu_{15c} + \nu_{15d} + \nu_{21a} + \nu_{21b} = 1\end{aligned}$$

For the purpose of this paper and due to the complexity of certain orders, we shall consider elements of order 2, 3, 7, 13, 14, 26, 35, 39, 65 and 91. We shall now consider each case separately.

Case (i). Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{2b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(\gamma_1 + 65) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(-\gamma_1 + 65) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{2}(\gamma_2 + 78) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{2}(-\gamma_2 + 6\nu_{2b} + 78) \geq 0$$

where $\gamma_1 = \nu_{2a} + 5\nu_{2b}$ and $\gamma_2 = 14\nu_{2a} - 6\nu_{2b}$. Clearly $\gamma \in \{1 + 2k \mid -33 \leq k \leq 32\}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in part (iii) of Theorem 2.

Case (ii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. Using Propositions 2 & 3,

$$\nu_{3a} + \nu_{3b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{3}(2\gamma + 65) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{3}(-\gamma + 65) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \tfrac{1}{3}(-6\nu_{3a} + 6) \geq 0$$

where $\gamma = 20\nu_{3a} - \nu_{3b}$. Clearly $\gamma \in \{2+3k \mid -11 \le k \le 21\}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are listed in part (iii) of Theorem 2.

Case (iii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 7$. By Proposition 2, $\nu_{kx} = 0$ for all

$$\begin{aligned} kx \in \{&2a, 2b, 3a, 3b, 4a, 4b, 4c, 5a, 5b, 5c, 5d, 6a, 6b, 8a, 8b, 10a, 10b, \\ &10c, 10d, 12a, 12b, 12c, 13a, 13b, 15a, 15b, 15c, 15d, 21a, 21b\}. \end{aligned}$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (iv). Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. Using Propositions 2 & 3,

$$\nu_{13a} + \nu_{13b} = 1.$$

Applying Proposition 5, we obtain the following system of inequalities:

$$\mu_1(u, \chi_2, 2) = \tfrac{1}{13}(\gamma_1 + 6) \ge 0; \quad \mu_1(u, \chi_{12}, 2) = \tfrac{1}{13}(-\gamma_1 + 384) \ge 0;$$
$$\mu_2(u, \chi_2, 2) = \tfrac{1}{13}(\gamma_2 + 6) \ge 0$$

where $\gamma_1 = 7\nu_{13a} - 6\nu_{13b}$ and $\gamma_2 = -6\nu_{13a} + 7\nu_{13b}$. Clearly $\gamma_1 \in \{7+13k \mid -1 \le k \le 29\}$. It follows that the only possible integer solutions for (ν_{13a}, ν_{13b}) are $(1, 0)$ and $(0, 1)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 4.

Case (v). Let $u \in V(\mathbb{Z}G)$ where $|u| = 14$. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{2b} + \nu_{7a} = 1.$$

Let $\gamma_1 = \nu_{2a} + 5\nu_{2b} + 2\nu_{7a}$, $\gamma_2 = 14\nu_{2a} - 6\nu_{2b} + \nu_{7a}$, $\gamma_3 = 20\nu_{2a} + \nu_{7a}$ and $\gamma_4 = 3\nu_{2a} + \nu_{2b}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 7\}$:

- $\chi(u^7) = 4\chi(2a) - 3\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6\gamma_1 + 66) \ge 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{14}(\gamma_1 + 74) \ge 0;$$
$$\mu_7(u, \chi_2, *) = \tfrac{1}{14}(-\gamma_1 + 88) \ge 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{14}(\gamma_2 + 3) \ge 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6\gamma_2 + 10) \ge 0.$$

Clearly $\gamma_1 \in \{-4, 10\}$ and $\gamma_2 \in \{-3\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{7a})$.

- $\chi(u^7) = -3\chi(2a) + 4\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6\gamma_1 + 94) \ge 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{14}(\gamma_1 + 46) \ge 0;$$
$$\mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6\gamma_1 + 60) \ge 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{14}(6\gamma_2 + 18) \ge 0;$$
$$\mu_2(u, \chi_3, *) = \tfrac{1}{14}(-\gamma_2 + 11) \ge 0.$$

Clearly $\gamma_1 \in \{-4, 10\}$ and $\gamma_2 \in \{-3, 11\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{7a})$.

- $\chi(u^7) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = \chi(7a)$ where $(m_1, m_2) \in \{(1, 0), (0, 1), (2, -1), (-1, 2)\}$.

Applying Proposition 5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{14}(6\gamma_1 + k_1) \ge 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{14}(\gamma_1 + k_2) \ge 0;$$
$$\mu_7(u, \chi_2, *) = \tfrac{1}{14}(-6\gamma_1 + k_3) \ge 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{14}(6\gamma_2 + k_4) \ge 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{14}(\gamma_2 + k_5) \ge 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{14}(-6\gamma_2 + k_6) \ge 0$$

where the values for k_i's and corresponding (m_1, m_2) values are as follows:

(m_1, m_2)	$(k_1, k_2, k_3, k_4, k_5, k_6)$
$(1, 0)$	$(78, 62, 76, 98, 63, 70)$
$(0, 1)$	$(82, 58, 72, 78, 83, 90)$
$(2, -1)$	$(74, 66, 80, 118, 43, 50)$
$(-1, 2)$	$(86, 54, 68, 58, 103, 110)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{7a})$ in all cases.

- $\chi(u^7) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6\gamma_1+70) \geq 0; \quad \mu_1(u,\chi_2,*) = \tfrac{1}{14}(\gamma_1+70) \geq 0;$$
$$\mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6\gamma_1+84) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{14}(6\gamma_2+138) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{14}(14\gamma_2+23) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6\gamma_2+30) \geq 0;$$
$$\mu_0(u,\chi_4,*) = \tfrac{1}{14}(-6\gamma_3+234) \geq 0.$$

Clearly $\gamma_1 \in \{0, 14\}$ and $\gamma_2 \in \{-23, -9, 5\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{7a})$.

- $\chi(u^7) = -2\chi(2a) + 3\chi(2b)$ and $\chi(u^2) = \chi(7a)$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(6\gamma_1+90) \geq 0; \quad \mu_1(u,\chi_2,*) = \tfrac{1}{14}(\gamma_1+50) \geq 0;$$
$$\mu_7(u,\chi_2,*) = \tfrac{1}{14}(-6\gamma_1+64) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{14}(6\gamma_2+38) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{14}(\gamma_2+123) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{14}(-6\gamma_2+130) \geq 0;$$
$$\mu_0(u,\chi_6,*) = \tfrac{1}{14}(60\gamma_4+320) \geq 0.$$

Clearly $\gamma_1 \in \{-8, 6\}$ and $\gamma_2 \in \{3, 17\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{7a})$.

Case (vi). Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 2 & 3,

$$\nu_{2a} + \nu_{2b} + \nu_{13a} + \nu_{13b} = 1.$$

Let $\gamma_1 = \nu_{2a} + 5\nu_{2b}$, $\gamma_2 = 7\nu_{2a} - 3\nu_{2b}$ and $\gamma_3 = 75\nu_{2a} + 15\nu_{2b} - 6\nu_{13a} + 7\nu_{13b}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 13\}$:

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1, m_2, m_3, m_4) \in \{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (2,-1,1,0), (2,-1,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{26}(12\gamma_1+k_1) \geq 0; \quad \mu_{13}(u,\chi_2,*) = \tfrac{1}{26}(-12\gamma_1+k_2) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \tfrac{1}{26}(\gamma_1+k_2) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2, m_3, m_4) values are as follows:

(m_1, m_2, m_3, m_4)	(k_1, k_2)
$(1, 0, 1, 0)$	$(66, 64)$
$(1, 0, 0, 1)$	$(66, 64)$
$(0, 1, 1, 0)$	$(70, 60)$
$(0, 1, 0, 1)$	$(70, 60)$
$(2, -1, 1, 0)$	$(62, 68)$
$(2, -1, 0, 1)$	$(62, 68)$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a}, \nu_{13b})$ in all cases.

- $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1, m_2, m_3, m_4) \in \{(3,-2,1,0), (3,-2,0,1), (-1,2,1,0), (-1,2,0,1), (-2,3,1,0), (-2,3,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{26}(12\gamma_1+k_1) \geq 0; \quad \mu_{13}(u,\chi_2,*) = \tfrac{1}{26}(-12\gamma_1+k_2) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{26}(24\gamma_2+k_3) \geq 0; \quad \mu_{13}(u,\chi_3,*) = \tfrac{1}{26}(-24\gamma_2+k_4) \geq 0;$$
$$\mu_1(u,\chi_{30},*) = \tfrac{1}{26}(-\gamma_3+k_5) \geq 0; \quad \mu_4(u,\chi_{30},*) = \tfrac{1}{26}(\gamma_3+k_6) \geq 0$$

where the values for k_i's and corresponding (m_1, m_2, m_3, m_4) values are as follows:

(m_1,m_2,m_3,m_4)	$(k_1,k_2,k_3,k_4,k_5,k_6)$
$(3,-2,1,0)$	$(58,72,132,24,4927,4537)$
$(3,-2,0,1)$	$(58,72,132,24,4914,4524)$
$(-1,2,1,0)$	$(74,56,52,104,4687,4777)$
$(-1,2,0,1)$	$(74,56,52,104,4674,4764)$
$(-2,3,1,0)$	$(78,52,32,124,4627,4837)$
$(-2,3,0,1)$	$(78,52,32,124,4614,4824)$

It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$ in all cases.

• $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1,m_2,m_3,m_4) \in \{(4,-3,1,0),(4,-3,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{26}(12\gamma_1 + k_1) \geq 0; \quad \mu_{13}(u,\chi_2,*) = \tfrac{1}{26}(-12\gamma_1 + k_2) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{26}(2\gamma_2 + k_3) \geq 0; \quad \mu_{13}(u,\chi_3,*) = \tfrac{1}{26}(-24\gamma_2 + k_4) \geq 0;$$
$$\mu_1(u,\chi_{30},*) = \tfrac{1}{26}(-\gamma_3 + k_5) \geq 0; \quad \mu_4(u,\chi_{30},*) = \tfrac{1}{26}(\gamma_3 + k_6) \geq 0$$

where the values for k_i's and corresponding (m_1,m_2,m_3,m_4) values are as follows:

(m_1,m_2,m_3,m_4)	$(k_1,k_2,k_3,k_4,k_5,k_6)$
$(4,-3,1,0)$	$(54,76,4,4,4987,4477)$
$(4,-3,0,1)$	$(54,76,4,4,4974,4464)$

It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$ in all cases.

• $\chi(u^{13}) = m_1\chi(2a) + m_2\chi(2b)$ and $\chi(u^2) = m_1\chi(7a) + m_2\chi(7b)$ where $(m_1,m_2,m_3,m_4) \in \{(-3,4,1,0),(-3,4,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{26}(12\gamma_1 + k_1) \geq 0; \quad \mu_{13}(u,\chi_2,*) = \tfrac{1}{26}(-12\gamma_1 + k_2) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{26}(24\gamma_2 + k_3) \geq 0; \quad \mu_2(u,\chi_3,*) = \tfrac{1}{26}(-2\gamma_2 + k_4) \geq 0;$$
$$\mu_1(u,\chi_{30},*) = \tfrac{1}{26}(-\gamma_3 + k_5) \geq 0; \quad \mu_4(u,\chi_{30},*) = \tfrac{1}{26}(\gamma_3 + k_6) \geq 0$$

where the values for k_i's and corresponding (m_1,m_2,m_3,m_4) values are as follows:

(m_1,m_2,m_3,m_4)	$(k_1,k_2,k_3,k_4,k_5,k_6)$
$(-3,4,1,0)$	$(82,48,12,12,4567,4897)$
$(-3,4,0,1)$	$(82,48,12,12,4554,4884)$

It follows that there are no possible integer solutions for $(\nu_{2a},\nu_{2b},\nu_{13a},\nu_{13b})$ in all cases.

Case (vii). Let $u \in V(\mathbb{Z}G)$ where $|u| = 35$. Using Propositions 2 & 3,

$$\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} + \nu_{7a} = 1.$$

Let $\gamma_1 = 5\nu_{5c} + 5\nu_{5d} + 2\nu_{7a}$, $\gamma_2 = 3\nu_{5a} + 3\nu_{5b} + 3\nu_{5c} + 3\nu_{5d} + \nu_{7a}$, $\gamma_3 = 9\nu_{5a} - 11\nu_{5b} - 6\nu_{5c} + 9\nu_{5d}$ and $\gamma_4 = 6\nu_{5a} - 9\nu_{5b} + 11\nu_{5c} - 9\nu_{5d}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{5,7\}$:

• $\chi(u^7) = m_1\chi(5a)+m_2\chi(5b)+m_3\chi(5c)+m_4\chi(5d)$ and $\chi(u^5) = \chi(7a)$ where $(m_1,m_2,m_3,m_4) \in \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,0,-1,1),(0,0,-1,2),(0,0,2,-1),(0,1,1,-1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{35}(24\gamma_1 + k_1) \geq 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{35}(-6\gamma_1 + k_2) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{35}(24\gamma_2 + k_3) \geq 0; \quad \mu_7(u,\chi_3,*) = \tfrac{1}{35}(-6\gamma_2 + k_4) \geq 0;$$
$$\mu_1(u,\chi_{11},*) = \tfrac{1}{35}(\gamma_3 + k_5) \geq 0; \quad \mu_{14}(u,\chi_{11},*) = \tfrac{1}{35}(-6\gamma_3 + k_6) \geq 0;$$
$$\mu_1(u,\chi_{13},*) = \tfrac{1}{35}(-\gamma_4 + k_7) \geq 0; \quad \mu_{14}(u,\chi_{13},*) = \tfrac{1}{35}(6\gamma_4 + k_8) \geq 0$$

where the values for k_i's and corresponding (m_1,m_2,m_3,m_4) values are as follows:

(m_1,m_2,m_3,m_4)	$(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)$
$(1,0,0,0)$	$(77,77,96,81,830,830,810,810)$
$(0,1,0,0)$	$(77,77,96,81,810,810,825,825)$
$(0,0,1,0)$	$(97,72,96,81,810,810,810,810)$
$(0,0,0,1)$	$(97,72,96,81,825,825,830,830)$
$(1,0,-1,1)$	$(77,77,96,81,845,845,830,830)$
$(0,0,-1,2)$	$(97,72,96,81,840,840,850,850)$
$(0,0,2,-1)$	$(97,72,96,81,795,795,790,790)$
$(0,1,1,-1)$	$(77,77,96,81,795,795,805,805)$

It follows that there are no possible integer solutions for $(\nu_{5a},\nu_{5b},\nu_{5c},\nu_{5d},\nu_{7a})$ in all cases.

- $\chi(u^7)=m_1\chi(5a)+m_2\chi(5b)+m_3\chi(5c)+m_4\chi(5d)$ and $\chi(u^5)=\chi(7a)$ where $(m_1,m_2,m_3,m_4)\in\{(1,1,-1,0),(1,1,0,-1)\}$.

Applying Proposition 5, we obtain:

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{35}(24\gamma_1+k_1)\geq 0; \quad \mu_5(u,\chi_2,*)=\tfrac{1}{35}(-4\gamma_1+k_2)\geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{35}(24\gamma_2+k_3)\geq 0; \quad \mu_7(u,\chi_3,*)=\tfrac{1}{35}(-6\gamma_2+k_4)\geq 0;\\
\mu_1(u,\chi_{11},*) &= \tfrac{1}{35}(\gamma_3+k_5)\geq 0; \quad \mu_{14}(u,\chi_{11},*)=\tfrac{1}{35}(-6\gamma_3+k_6)\geq 0;\\
\mu_1(u,\chi_{13},*) &= \tfrac{1}{35}(-\gamma_4+k_7)\geq 0; \quad \mu_{14}(u,\chi_{13},*)=\tfrac{1}{35}(6\gamma_4+k_8)\geq 0
\end{aligned}$$

where the values for k_i's and corresponding (m_1,m_2,m_3,m_4) values are as follows:

(m_1,m_2,m_3,m_4)	$(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8)$
$(1,1,-1,0)$	$(57,43,96,81,830,830,825,825)$
$(1,1,0,-1)$	$(57,43,96,81,815,815,805,805)$

It follows that there are no possible integer solutions for $(\nu_{5a},\nu_{5b},\nu_{5c},\nu_{5d},\nu_{7a})$ in all cases.

Case $(viii)$. Let $u\in V(\mathbb{Z}G)$ where $|u|=39$. Using Propositions 2 & 3,

$$\nu_{3a}+\nu_{3b}+\nu_{13a}+\nu_{13b}=1.$$

Let $\gamma=20\nu_{3a}-\nu_{3b}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n\in\{3,13\}$:

- $\chi(u^{13})=m_1\chi(3a)+m_2\chi(3b)$ and $\chi(u^3)=m_3\chi(13a)+m_4\chi(13b)$ where $(m_1,m_2,m_3,m_4)\in\{(1,0,1,0),(1,0,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{39}(24\gamma+105)\geq 0; \quad \mu_{13}(u,\chi_2,*)=\tfrac{1}{39}(-12\gamma+45)\geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{3a},\nu_{3b},\nu_{13a},\nu_{13b})$ in all cases.

- $\chi(u^{13})=m_1\chi(3a)+m_2\chi(3b)$ and $\chi(u^3)=m_3\chi(13a)+m_4\chi(13b)$ where $(m_1,m_2,m_3,m_4)\in\{(0,1,1,0),(0,1,0,1),(-1,2,1,0),(-1,2,0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{39}(24\gamma+k_1)\geq 0; \quad \mu_{13}(u,\chi_2,*)=\tfrac{1}{39}(-12\gamma+k_2)\geq 0;$$
$$\mu_1(u,\chi_2,*)=\tfrac{1}{39}(\gamma+k_3)\geq 0$$

where the values for k_i's and corresponding (m_1,m_2,m_3,m_4) values are as follows:

(m_1,m_2,m_3,m_4)	(k_1,k_2,k_3)
$(0,1,1,0)$	$(63,66,66)$
$(0,1,0,1)$	$(63,66,66)$
$(-1,2,1,0)$	$(21,87,87)$
$(-1,2,0,1)$	$(21,87,87)$

It follows that there are no possible integer solutions for $(\nu_{3a},\nu_{3b},\nu_{13a},\nu_{13b})$ in all cases.

Case (ix). Let $u\in V(\mathbb{Z}G)$ where $|u|=65$. Using Propositions 2 & 3,

$$\nu_{5a}+\nu_{5b}+\nu_{5c}+\nu_{5d}+\nu_{13a}+\nu_{13b}=1.$$

Let $\gamma_1=\nu_{5a}+\nu_{5b}+\nu_{5c}+\nu_{5d}$ and $\gamma_2=\nu_{5c}+\nu_{5d}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n\in\{5,13\}$:

- $\chi(u^{13})=m_1\chi(5a)+m_2\chi(5b)+m_3\chi(5c)+m_4\chi(5d)$ and $\chi(u^5)=m_5\chi(13a)+m_6\chi(13b)$ where $(m_1,m_2,m_3,m_4,m_5,m_6)\in\{(1,0,0,0,1,0),(1,0,0,0,0,1),(0,1,0,0,1,0),(0,1,0,0,0,1),$ $(0,0,1,0,1,0),(0,0,1,0,0,1),(0,0,0,1,1,0),(0,0,0,1,0,1),(1,0,-1,1,1,0),(1,0,-1,1,0,1),$ $(0,0,-1,2,1,0),(0,0,-1,2,0,1),(0,0,2,-1,1,0),(0,0,2,-1,1,0),(0,1,1,-1,1,0),(0,1,1,-1,0,1)\}$.

Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_3,*)=\tfrac{1}{65}(144\gamma_1+90)\geq 0;\quad \mu_{13}(u,\chi_3,*)=\tfrac{1}{65}(-36\gamma_1+75)\geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{5a},\nu_{5b},\nu_{5c},\nu_{5d},\nu_{13a},\nu_{13b})$ in all cases.

- $\chi(u^{13})=m_1\chi(5a)+m_2\chi(5b)+m_3\chi(5c)+m_4\chi(5d)$ and $\chi(u^5)=m_5\chi(13a)+m_6\chi(13b)$ where $(m_1,m_2,m_3,m_4,m_5,m_6)\in\{(1,1,-1,0,1,0),(1,1,-1,0,0,1),(1,1,0,-1,1,0),(1,1,0,-1,0,1)\}$.

Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{65}(240\gamma_2+45)\geq 0;\quad \mu_{13}(u,\chi_2,*)=\tfrac{1}{65}(-60\gamma_2+70)\geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{5a},\nu_{5b},\nu_{5c},\nu_{5d},\nu_{13a},\nu_{13b})$ in all cases.

Case (x). Let $u\in V(\mathbb{Z}G)$ where $|u|=91$. Using Propositions 2 & 3,

$$\nu_{7a}+\nu_{13a}+\nu_{13b}=1.$$

Consider the cases $\chi(u^{13})=\chi(7a)$ and $\chi(u^7)=m_1\chi(13a)+m_1\chi(13b)$ where $(m_1,m_2)\in\{(1,0),(0,1)\}$. Applying Proposition 5, we obtain:

$$\mu_0(u,\chi_2,*)=\tfrac{1}{91}(144\nu_{7a}+77)\geq 0;\quad \mu_{13}(u,\chi_2,*)=\tfrac{1}{91}(-24\nu_{7a}+63)\geq 0.$$

It follows that there are no possible integer solutions for $(\nu_{7a},\nu_{13a},\nu_{13b})$.

We shall now consider the prime graph of $G=G_2(4)$. G contains elements of order 6, 10, 15 and 21. Therefore $[2,3]$, $[2,5]$, $[3,5]$ and $[3,7]$ are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. Clearly $\pi(G)=\pi(V(\mathbb{Z}G))$, since there are no torsion units of order 14, 26, 35, 39, 65, and 91 in $V(\mathbb{Z}G)$. This completes the proof.

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